Epsilon-Delta Definitions of Limits: Intuition and Examples

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Abstract

Limits are the basis of differentiation and integration — the two fundamental operations of Calculus. In this summary we will explore formal epsilon-delta definitions for different types of limits as well as their intuitive meanings and verification processes

Keywords: math, calculus, limits

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Epsilon-Delta Definitions of Limits: Intuition and Examples

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1 Introduction

Limits are the basis of differentiation and integration — the two fundamental operations of Calculus. In this summary we will explore formal epsilon-delta definitions for different types of limits as well as their intuitive meanings and verification processes.

2 Finite Limits

Definition. We say the limit $\lim_{x\to c} f(x) = L$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x) - L| < \epsilon$.

A wordless way of saying this is:

$$\lim_{x\to c} f(x) = L \iff \forall\, \epsilon > 0, \exists\, \delta > 0 \;\; s.t.\;\; 0 < |x-c| < \delta \implies |f(x)-L| < \epsilon.$$

where L and c are real numbers. Intuitively, this means that, for the limit to exist, we must be able to find an arbitrarily small distance δ between x and c that guarantees that $f(x)$ and L are less than an arbitrarily small distance ϵ apart.

Therefore, in order to verify a finite limit, we must find a δ such that $|f(x) - L| < \epsilon$. We can write δ in terms of ϵ in order to satisfy the inequality for every value of ϵ. We explore what this means in the following example.

Example 1. Prove that $\lim_{x\to -1}(4x+1) = -3$.

Proof. In order to write δ in terms of ϵ, we start by writing out $|f(x) - L| < \epsilon$:

$$\begin{aligned} |f(x) - L| < \epsilon &\implies |4x+1-(-3)| < \epsilon \\ &\implies |4x+4| < \epsilon \\ &\implies 4|x+1| < \epsilon \\ &\implies |x+1| < \epsilon/4. \end{aligned}$$

Our inequality is now in the form $|x-c| < \delta$. We can now choose $\delta = \epsilon/4$ and substitute it back into $|x-c| < \delta$:

$$\begin{aligned} |x-c| < \delta &\implies |x-(-1)| < \epsilon/4 \\ &\implies |x+1| < \epsilon/4 \\ &\implies 4|x+1| < \epsilon \\ &\implies |4x+4| < \epsilon \\ &\implies |4x+1-(-3)| < \epsilon \\ &\implies |f(x)-L| < \epsilon. \end{aligned}$$

We can choose a δ such that $0 < |x-c| < \delta \implies |f(x)-L| < \epsilon$. Therefore, $\lim_{x\to -1}(4x+1) = -3$. □

δ is Not Unique

Example. Prove that $\lim_{x \to 3} x^2 = 9$.

Proof. We are looking for a value of δ such that

$$|x-3| < \delta \implies |x^2 - 9| < \epsilon.$$

We can factor $|x^2 - 9|$ as the difference of two squares to rewrite the statement as

$$\begin{aligned} |x-3| < \delta &\implies |x+3||x-3| < \epsilon \\ &\implies \delta|x+3| < \epsilon. \end{aligned}$$

By bounding the factor $|x+3|$ to a constant, we can express δ solely in terms of ϵ. We make the restriction[1] $\delta \le 1$. It follows that

$$\begin{aligned} |x-3| < 1 &\implies -1 < x+3 < 1 \\ &\implies 5 < x+3 < 7 \\ &\implies |x+3| < 7. \end{aligned}$$

Therefore,

$$|x-3||x+3| < 7\delta < \epsilon$$

and it follows that we can choose $\delta = \epsilon/7$.

However, since we made the restriction $\delta \le 1$, we must account for when $\epsilon > 7$, as that would make $\delta > 1$. Thus, we choose δ to be the minimum of 1 and $\epsilon/7$, or $\delta = \min(1, \epsilon/7)$. To verify the limit, we assume that $0 < |x-3| < \delta$:

$$\begin{aligned} |x-3| < \delta &\implies |x-3||x+3| < \delta|x+3| \\ &\implies |x-3||x+3| < 7\delta \\ &\implies |x^2-9| < 7 \cdot \frac{\epsilon}{7} \\ &\implies |x^2-9| < \epsilon. \end{aligned}$$

We can choose a δ such that $0 < |x-c| < \delta \implies |f(x) - L| < \epsilon$. Therefore, $\lim_{x \to 3} x^2 = 9$. □

3 Infinite Limits

Definition. We say the limit $\lim_{x \to c} f(x) = \infty$[2] if, for any $M > 0$, there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $f(x) > M$.

Intuitively, we are saying that, for a limit to approach infinity, $f(x)$ must be large than an arbitrarily large value M when x and c are an arbitrarily small distance δ apart.

Similar to verifying a finite limit, we verify an infinite limit by choosing a δ such that $f(x) > M$. Just like how we wrote δ in terms of ϵ above, in the following examples we will write δ in terms of M.

Example 2. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

[1] We can pick any constant as an upper bound for δ, but we use 1 for convenience.

[2] Strictly speaking, we cannot say $\lim_{x \to c} f(x) = \infty$, since infinity cannot be equal to anything. A more mathematically sound way of expressing the value of the limit would be to say that $f(x)$ increases without bounds, or that it approaches infinity.

Proof. In order to write δ in terms of M, we start by writing out $f(x) > M$:

$$\begin{aligned} f(x) > M &\implies \frac{1}{x^2} > M \\ &\implies \frac{1}{M} > x^2 \\ &\implies \sqrt{\frac{1}{M}} > x \\ &\implies x < \frac{1}{\sqrt{M}}. \end{aligned}$$

Our inequality is now in the form $|x - c| < \delta$, so we choose $\delta = \frac{1}{\sqrt{M}}$.

To verify the limit, we assume that $0 < |x - c| < \delta$, and substitute our choice of $\delta = \frac{1}{\sqrt{M}}$ into the inequality:

$$\begin{aligned} 0 < |x - c| < \delta &\implies |x| < \delta \\ &\implies x^2 < \delta^2 \\ &\implies \frac{1}{x^2} > \frac{1}{\delta^2} \\ &\implies \frac{1}{x^2} > \frac{1}{(\frac{1}{\sqrt{M}})^2} \\ &\implies \frac{1}{x^2} > \frac{1}{\frac{1}{M}} \\ &\implies \frac{1}{x^2} > M \\ &\implies f(x) > M. \end{aligned}$$

We can choose a δ such that $0 < |x - c| < \delta \implies f(x) > M$. Therefore, $\lim_{x \to 0} \frac{1}{x^2} = \infty$. □

Example 3. Prove that $\lim_{x \to 1} \frac{3}{(x-1)^2} = \infty$.

Proof. We once again start by writing out $f(x) > M$ in order to write δ in terms of M:

$$\begin{aligned} f(x) > M &\implies \frac{3}{(x-1)^2} > M \\ &\implies \frac{3}{M} > (x-1)^2 \\ &\implies \sqrt{\frac{3}{M}} > x - 1 \\ &\implies x - 1 < \sqrt{\frac{3}{M}}. \end{aligned}$$

Our inequality is now in the form $|x - c| < \delta$, so we choose $\delta = \sqrt{\frac{3}{M}}$.

We assume that $0 < |x - c| < \delta$, and substitute our choice of $\delta = \sqrt{\frac{3}{M}}$ into the inequality:

$$
\begin{aligned}
0 < |x - c| < \delta &\implies |x - 1| < \delta \\
&\implies (x-1)^2 < \delta^2 \\
&\implies \frac{1}{(x-1)^2} > \frac{1}{\delta^2} \\
&\implies \frac{1}{(x-1)^2} > \frac{1}{(\sqrt{\frac{3}{M}})^2} \\
&\implies \frac{1}{(x-1)^2} > \frac{1}{\frac{3}{M}} \\
&\implies \frac{1}{(x-1)^2} > \frac{M}{3} \\
&\implies \frac{3}{(x-1)^2} > M \\
&\implies f(x) > M.
\end{aligned}
$$

We can choose a δ such that $0 < |x - c| < \delta \implies f(x) > M$. Therefore, $\lim_{x \to 1} \frac{3}{(x-1)^2} = \infty$. □

4 Limits at Infinity

Definition. We say that $\lim_{x \to \infty} f(x) = L$ if, for any $\epsilon > 0$, there exists an $M > 0$ such that, if $x > M$, then $|f(x) - L| < \epsilon$.

Intuitively, we say that the limit of a function exists at infinity if the distance between $f(x)$ and L is less than an arbitrarily small distance ϵ when x is greater than an arbitrarily large value M.

To verify limits at infinity, we start with $|f(x) - L| < \epsilon$ and rearrange the inequality algebraically until it fits the form $x > M$.

Example 4. Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Proof. We start with $|f(x) - L| < \epsilon$:

$$
\begin{aligned}
|f(x) - L| < \epsilon &\implies \left| \frac{1}{x} - 0 \right| < \epsilon \\
&\implies \frac{1}{\epsilon} < x.
\end{aligned}
$$

Now that our inequality is in the form $x > M$, we can choose $M = \frac{1}{\epsilon}$ [3].

We assume $x > M > 0$ and substitute $M = \frac{1}{\epsilon}$ into the inequality:

$$
\begin{aligned}
x > M &\implies x > \frac{1}{\epsilon} \\
&\implies \frac{1}{x} < \epsilon \\
&\implies \left| \frac{1}{x} - 0 \right| < \epsilon \\
&\implies |f(x) - L| < \epsilon.
\end{aligned}
$$

We can choose an M such that $x > M \implies |f(x) - L| < \epsilon$. Therefore, $\lim_{x \to \infty} \frac{1}{x} = 0$. □

Example 5. Prove that $\lim_{x \to \infty} \frac{6x+2}{2x+8} = 3$.

[3] Since ϵ is an infinitesimal, $\frac{1}{\epsilon}$ is a value greater than any real number, which makes sense since we intuitively interpret M as an arbitrarily large value.

Proof. We start with $|f(x) - L| < \epsilon$:

$$\begin{aligned}|f(x)-L|<\epsilon &\implies \left|\frac{6x+2}{2x+8}-3\right|<\epsilon\\ &\implies \left|\frac{6x+2}{2x+8}-\frac{3(2x+8)}{2x+8}\right|<\epsilon\\ &\implies \left|\frac{6x+2-6x-24}{2x+8}\right|<\epsilon\\ &\implies \left|\frac{-22}{2x+8}\right|<\epsilon.\end{aligned}$$

Since we want $x > M > 0$, we can assume $x > 0$ to drop the absolute value.

$$\begin{aligned}\frac{22}{2x+8}<\epsilon &\implies 22<\epsilon(2x+8)\\ &\implies 22<2\epsilon x+8\epsilon\\ &\implies 22-8\epsilon<2\epsilon x\\ &\implies \frac{22-8\epsilon}{2\epsilon}<x.\end{aligned}$$

We can choose $M = \frac{22-8\epsilon}{2\epsilon}$. We now assume[4] $x > M = \frac{22-8\epsilon}{2\epsilon}$ to prove that $|f(x) - L| < \epsilon$.

$$\begin{aligned}x>M &\implies x>\frac{22-8\epsilon}{2\epsilon}\\ &\implies 2\epsilon x>22-8\epsilon\\ &\implies 2\epsilon x+8\epsilon>22\\ &\implies \epsilon(2x+8)>22\\ &\implies \epsilon>\frac{22}{2x+8}\\ &\implies \left|\frac{-22}{2x+8}\right|<\epsilon\\ &\implies \left|\frac{6x+2}{2x+8}-3\right|<\epsilon\\ &\implies |f(x)-L|<\epsilon.\end{aligned}$$

We can choose an M such that $x > M \implies |f(x) - L| < \epsilon$. Therefore, $\lim_{x\to\infty} \frac{6x+2}{2x+8} = 3$. □

5 Infinite Limits at Infinity

Definition. We say the limit $\lim_{x\to\infty} f(x) = \infty$ if, for any $M > 0$, there exists an $N > 0$ such that if $x > N$, then $f(x) > M$.

We can combine the definitions for infinite limits and limits at infinity into a definition for a limit that increases without bound as the independent variable increases without bound. Intuitively, if $f(x)$ is greater than an arbitrarily large value M when x is greater than an arbitrarily large value N, $f(x)$ has an infinite limit at infinity.

To verify these types of limits, we start with $f(x) > M$ and rearrange the inequality into the form $x > N$.

Example 6. Prove that $\lim_{x\to\infty}(4x + 1) = \infty$.

[4] We once again observe that $\frac{22-8\epsilon}{2\epsilon}$ approaches infinity since ϵ is an infinitesimal.

Proof. We start with $f(x) > M$:

$$\begin{aligned} f(x) > M &\implies 4x + 1 > M \\ &\implies 4x > M - 1 \\ &\implies x > \frac{M-1}{4}. \end{aligned}$$

Therefore, we choose $N = \frac{M-1}{4}$. We assume $x > N$ and substitute $N = \frac{M-1}{4}$ into the inequality:

$$\begin{aligned} x > N &\implies x > \frac{M-1}{4} \\ &\implies 4x > M - 1 \\ &\implies 4x + 1 > M \\ &\implies f(x) > M. \end{aligned}$$

We can choose an N such that $x > N \implies f(x) > M$. Therefore, $\lim_{x\to\infty}(4x+1) = \infty$. □

Example 7. Prove that $\lim_{x\to\infty} \frac{11x+3}{7} = \infty$.

Proof. We start with $f(x) > M$:

$$\begin{aligned} f(x) > M &\implies \frac{11x+3}{7} > M \\ &\implies 11x + 3 > 7M \\ &\implies 11x > 7M - 3 \\ &\implies x > \frac{7M-3}{11}. \end{aligned}$$

Therefore, we choose $N = \frac{7M-3}{11}$. We assume $x > N$ and substitute $N = \frac{7M-3}{11}$ into the inequality:

$$\begin{aligned} x > N &\implies x > \frac{7M-3}{11} \\ &\implies 11x > 7M - 3 \\ &\implies 11x + 3 > 7M \\ &\implies \frac{11x+3}{7} > M \\ &\implies f(x) > M. \end{aligned}$$

We can choose an N such that $x > N \implies f(x) > M$. Therefore, $\lim_{x\to\infty} \frac{11x+3}{7} = \infty$. □

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