Mathematical Insights into Planck’s Law and Its Physical Implications

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Abstract

This paper explores the mathematical underpinnings of Planck's law, a fundamental principle in quantum mechanics that describes the radiation emitted by a black body. Through a series of derivations, we demonstrate how the law leads to Stefan-Boltzmann’s law, which relates the total energy emitted by a black body to its temperature. We provide a detailed mathematical interpretation of the integrals and functions arising from Planck’s formula, specifically focusing on the Gamma and Riemann Zeta functions. By connecting physical principles with advanced mathematical concepts, we highlight the significance of these functions in both theoretical physics and mathematics. Additionally, we propose two thought-provoking problems involving dominoes and a crawling beetle to further engage with the concepts discussed.

Keywords: Planck's law, Stefan-Boltzmann law, black body radiation, Gamma function, Riemann Zeta function, quantum mechanics, mathematical derivations, theoretical physics, energy emission.

1. Planck’s Law

Planck’s law is one of the fundamental principles of modern quantum mechanics. It connects the temperature in the model of a *black body* with the energy it radiates. Any object with a non-zero temperature emits radiation — even we emit light, though in the infrared spectrum.

It is due to Planck’s law that metal first turns dark red, then gets brighter, reaching white heat when heated. It also underpins the operation of remote thermometers — they read the energy emitted by us in the infrared spectrum and then calculate our temperature. Planck’s law also accurately describes the radiation of stars, planets, satellites, and much more.

In one form, the law can be written as:

$$B(\nu,T)=\frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1},$$

where $B(\nu, T)$ is the spectral energy density of a black body at temperature T and frequency ν, h is Planck’s constant, c is the speed of light, and k is Boltzmann’s constant.

2. Corollaries

2.1 Physical Interpretation

Let’s consider the total energy flux of blackbody radiation, which can be represented as the integral of energy flux densities over all wave frequencies:

$$B(T)=\int\limits_0^\infty B(\nu,T)d\nu=\int\limits_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$$

Let’s make a substitution $x=\frac{h\nu}{kT}$, then $\nu=\frac{kT}{h}x$ and $d\nu=\frac{kT}{h}dx$. Substituting this x, we get:

$$\int\limits_0^\infty \frac{\frac{2h}{c^2}\left(\frac{kT}{h}\right)^3x^3\frac{kT}{h}dx}{e^x-1}=\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\int\limits_0^\infty\frac{x^3}{e^x-1}dx.$$

From this, we essentially get Stefan-Boltzmann’s law:

$$B(T) = \left(\frac{2k^4}{c^2h^3}\int_0^\infty \frac{x^3}{e^x-1}dx\right)T^4$$

In Stefan-Boltzmann's law, the energy flux density is π times greater than in this case than the energy brightness:

$$F = \pi B(T)$$

It's also interesting to consider the average energy of a blackbody photon $\langle \epsilon_\gamma \rangle$. By definition, it equals $\langle \epsilon_\gamma \rangle = u/\langle n_\gamma \rangle$, where u is the energy density and $\langle n_\gamma \rangle$ is the average photon concentration.

Energy density of blackbody radiation:

$$u = \frac{1}{c}\int_0^\infty B(\nu, T)d\nu d\Omega =$$

$$= \frac{4\pi}{c}\int_0^\infty B(\nu, T)d\nu = \frac{4\pi}{c}B(T).$$

Average concentration of blackbody photons:

$$\langle n_\gamma \rangle = \frac{4\pi}{c}\int_0^\infty \frac{B(\nu, T)}{h\nu}d\nu =$$

$$= \frac{4\pi}{c}\int_0^\infty \frac{2\nu^2}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}d\nu =$$

By making a similar substitution, we get:

$$= \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\int_0^\infty \frac{x^2}{e^x-1}dx.$$

Thus, the average photon concentration is:

$$\langle \epsilon_\gamma \rangle = \frac{\frac{4\pi}{c}\frac{2k^4}{c^2h^3}\int_0^\infty \frac{x^3}{e^x-1}dxT^4}{\frac{2k^3}{c^2h^3}\int_0^\infty \frac{x^2}{e^x-1}dxT^3} = kT\frac{\int_0^\infty \frac{x^3}{e^x-1}dx}{\int_0^\infty \frac{x^2}{e^x-1}dx}$$

2.2 Mathematical Interpretation

Let's consider the function $f(m)$:

$$f(m) = \int_0^\infty \frac{x^m}{e^x-1}dx$$

It's easy to notice that in the previous section, we needed to find $f(2)$ and $f(3)$. Having set this task, let's try to transform our $f(m)$ into a more familiar and convenient form for mathematicians. First, multiply the numerator and denominator of the fraction by e^{-x}:

$$\int_0^\infty \frac{x^m}{e^x-1}dx = \int_0^\infty \frac{e^{-x}x^m}{1-e^{-x}}dx$$

Now let's play with the denominator of the fraction. Notice that $e^{-x} < 1$ when $x\in(0, \infty]$, which, according to the formula for the sum of a decreasing geometric series, gives the following expression:

$$\frac{1}{1-e^{-x}} = 1 + (e^{-x}) + (e^{-x})^2 + (e^{-x})^3 + \ldots = \sum_{k=0}^\infty e^{-xk}$$

Substituting this identity, we get the integral of the sum:

$$f(m) = \int_0^\infty e^{-x}x^m\left(\sum_{k=0}^\infty e^{-xk}\right)dx = \int_0^\infty \left(\sum_{k=0}^\infty x^m e^{-x(k+1)}dx\right)$$

Definition 1. A sequence of functions $f_n(x): X\to R$ converges uniformly to a function $f(x)$ if for any $\epsilon > 0$, there exists $N > 0$ such that for any $n > N$ and $x \in X$:

$$|f_n(x) - f(x)| < \epsilon$$

Lemma 1. *A finite series can be expressed as the sum of definite integrals (interchanging the summation and integration signs):*

$$\int_a^b \sum_{n=c}^d f(n,x)dx = \sum_{n=c}^d \int_a^b f(n,x)dx,$$

where the limits of integration are $a, b\in R\cup(-\infty; \infty)$*, and the sums range from* $c, d \in Z_+ \cup (0; \infty)$*. If* $d = \infty$*, it is necessary that* $f(n, x) \geq 0$ *for all* $x \in [a, b]$*, and that the sequence of partial sums* $F_N(x) = \sum_{n=c}^{N} f(n, x)$ *converges uniformly (Definition 1) to the function* $f(x)$

Proof. This lemma is obvious for $d \in Z$, since the integral of the sum of a finite number of terms can be represented as the sum of the integrals of each term. When $d = \infty$, the lemma turns into the following fact:

$$\int_a^b \lim_{N\to\infty} F_N(x)dx = \lim_{N\to\infty} \int_a^b F_N(x)dx,$$

where

$$F_N(x) = \sum_{n=c}^{N} f(n, x).$$

This interchange is valid because the sum of the series is the limit of the sequence of its partial sums. Suppose $lim_{N\to\infty} F_N(x) = f(x)$. The fact that $f(x)$ is the limit of the sequence $F_N(x)$ means that for any $x \in [a, b]$ and $\epsilon > 0$, there exists M such that for any $N > M$:

$$f(x) - f_N(x) < \epsilon$$

We will show that for all $\epsilon > 0$, there exists $M > 0$, such that for any $N > M$:

$$\left| \int_a^b f(x)dx - \int_a^b F_N(x)dx \right| < \epsilon.$$

Note that $f(x) > F_N(x)$ for any $x \in [a, b]$, since the sequence $F_N(x)$ is monotonically increasing. Let's denote $g_n(x) = F_N(x) - f(x)$. Then the sequence $g_n(x)$ is non-increasing and converges uniformly to zero:

$$\int_a^b f(x)dx - \int_a^b F_N(x)dx \leq$$

$$\leq \int_a^b g_n(x)dx.$$

Suppose this is not true. That is, there exists an ϵ such that for any n:

$$\int_a^b g_n(x)dx = \delta \geq \epsilon.$$

Let's fix some n and consider the interval $[k, t] \subset [a, b]$ such that:

$$\int_k^t g_n(x)dx > \delta - \epsilon.$$

Therefore, on $[a, k) \cup (t, b]$ the integral is less than ϵ, and does not increase as N increases. Since $g_n(x)$ converges uniformly to zero, there exists an N such that $g_n(x) < \epsilon'/(t - k)$ for any $x \in [k, t]$, and then:

$$\int_k^t g_n(x)dx < \int_k^t \epsilon'/(t-k)dx = \epsilon'.$$

This leads to a contradiction, and therefore our lemma is proven. □

Lemma 2. *The sequence* $f_n(x) = \sum_{k=0}^{n} x^m e^{-x(k+1)}$ *converges uniformly on* $[0, +\infty)$

Proof. We will prove that for any $\epsilon > 0$, there exists N, such that for any $n > N$ and $x \in [0, +\infty)$:

$$\sum_{k=n}^{\infty} x^m e^{-x(k+1)} < \epsilon,$$

which will be equivalent to the required conditions. Fix n and x and transform slightly:

$$\sum_{k=n}^{\infty} x^m e^{-x(k+1)} < \epsilon \Leftrightarrow \qquad (\epsilon' = \frac{\epsilon}{x^m} > 0)$$

$$\Leftrightarrow \sum_{k=n+1}^{\infty} (e^{-x})^k < \epsilon' \Leftarrow \qquad (t = e^{-x})$$

$$\Leftarrow \sum_{k=N}^{\infty} t^k < \epsilon' \qquad (t \in (0, 1))$$

Thus, the necessary statement is equivalent to the fact that the geometric progression from a number less than 1 converges, which is true (to $\frac{1}{1-t}$). □

It is claimed that in this case, the integral and sum can be interchanged by Lemma *1*, since $x^m e^{-x(k+1)}$ is always positive and the series converges uniformly by Lemma *2*:

$$\int_0^\infty \left(\sum_{k=0}^\infty x^m e^{-x(k+1)} dx\right) =$$

$$= \sum_{k=0}^\infty \left(\int_0^\infty x^m e^{-x(k+1)} dx\right)$$

In each integral of this sum, let's replace x with $y_k = x(k+1)$. Then $dx = \frac{dy_k}{k+1}$. Substituting, we get:

$$f(m) = \sum_{k=0}^\infty \left(\int_0^\infty \frac{y_k^m}{(k+1)^m} e^{-y_k} \frac{dy_k}{k+1}\right) =$$

$$= \sum_{k=0}^\infty \left(\frac{1}{(k+1)^{m+1}} \int_0^\infty y_k^m e^{-y_k} dy_k\right) =$$

$$= \sum_{k=0}^\infty \frac{\Gamma(m+1)}{(k+1)^{(m+1)}} =$$

$$= \Gamma(m+1) \sum_{k=1}^\infty \frac{1}{k^{(m+1)}} =$$

$$= \Gamma(m+1)\zeta(m+1)$$

Where $\Gamma(m+1)$ and $\zeta(m+1)$ are the Gamma and Zeta functions of $m+1$, which are simply their definitions.

Thus, after some transformations, we have reduced a purely physical function associated with Planck's formula to the product of two known mathematical functions, and now we just need to deal with each of them separately:

$$\int_0^\infty \frac{x^m}{e^x - 1} dx = \Gamma(m+1)\zeta(m+1)$$

3. Gamma Function

Definition 2. The Gamma function is a generalization of factorials for arguments in C, introduced by Leonhard Euler. It plays an important role in a wide range of sciences, including astronomy.

$$\Gamma(m+1) = \int_0^\infty x^m e^{-x} dx$$

Lemma 3. *The following identity holds:*
$\Gamma(m+1) = m\Gamma(m)$

Proof. Let's integrate $\Gamma(m+1)$ by parts as follows:

$$\Gamma(m+1) = \int_0^\infty x^m e^{-x} dx =$$

$$= \underbrace{-\frac{x^m}{e^x}\bigg|_0^\infty}_{=0} + \int_0^\infty m x^{m-1} e^{-x} dx =$$

$$= m\Gamma(m)$$

□

Remark 1. Integration by parts was used here, specifically this formula:

$$\int u dv = uv - \int v du$$

It is easy to verify that the above transition can be obtained by substituting the integration limits from 0 to ∞ and the variables u *and* v as follows:

$$v = x^m \qquad dv = m x^{m-1} dx$$

$$du = e^{-x} dx \qquad u = -e^{-x}$$

Theorem 4. *For all* $m \in Z_+$, *the following relation holds:*

$$\Gamma(m) = (m-1)!$$

Proof. We will prove Theorem *4* by INDUCTION. BASE CASE: $m = 1$.

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx =$$

$$= -\left.\frac{1}{e^x}\right|_0^\infty = -(0-1) = 1$$

INDUCTIVE STEP: $m \to m+1$. By the inductive hypothesis, $\Gamma(m) = (m-1)!$, then by Lemma 3:

$$\Gamma(m+1) = m\Gamma(m) = m(m-1)! = m!$$

Thus, this identity holds for all $m \in Z_+$. □

4. Zeta Function

Definition 3. The Riemann Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

We have the opportunity to find the exact value for $\zeta(s)$ if s is an even number, but the zeta function for odd numbers cannot be calculated so easily.

4.1 For even m

The following representation was described by Leonhard Euler in "Introductio in analysin infinitorum", Chapter 9, §§155-158:

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right)$$

The proof of this equation can also be found there; however, we will not discuss it here. The curious reader may try to prove this on their own.

Let's write the Taylor series for the same function:

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}.$$

Thus, for all $x \in R/\{0\}$:

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}.$$

The coefficients for each power of x must match in both expansions. Otherwise, let's consider the lowest power x^k, which appears in their difference with a nonzero coefficient a. All other terms in this series become $o(x^k)$ as $x \to 0$, so:

$$0 = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right) - \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} =$$

$$= ax^k + o(x^k), \text{ as } x \to 0.$$

Thus, $0 = ax^k + o(x^k)$ as $x \to 0$, meaning that $a = 0$, a contradiction?!

In that case, let's find the coefficient for x^2 in the first expansion by expanding the terms. x^2 will only appear in monomials obtained from such products:

$$1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \cdot \left(-\frac{x^2}{\pi^2 n^2}\right) \cdot 1 \cdot \ldots \cdot 1 \cdot 1 \cdot 1 \cdot \ldots .$$

Thus, we have:

$$\sum_{1}^{\infty} -\frac{x^2}{\pi^2 n^2} = -\frac{x^2}{6}$$

$$\frac{1}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2} = \frac{1}{6}$$

$$\zeta(2) = \frac{\pi^2}{6}$$

The coefficients for x^4 match, so:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\pi^2 n^2}\sum_{m=n+1}^{\infty}\frac{1}{\pi^2 m^2}\right) = \frac{1}{120}$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right) - \frac{1}{n^4}\right) = \frac{\pi^4}{120}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\zeta(2)\right) - \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{60}$$

$$\zeta(2)^2 - \zeta(4) = \frac{\pi^4}{60}$$

$$\zeta(4) = \frac{\pi^4}{90}.$$

Remark 1. Similarly, $\zeta(2k)$ can be found for any natural k.

4.2 For odd m

As mentioned earlier, unfortunately, there is no simple representation for $\zeta(3)$, but for physical purposes, it is sufficient to use an approximately accurate value.

$$\zeta(3) = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \epsilon$$

We will bound $\epsilon = \sum_{6}^{\infty} \frac{1}{n^3}$ from above with an integral:

$$\epsilon = \int_{5}^{\infty} \frac{dx}{x^3} = -\frac{1}{2}\frac{1}{x^2}\Big|_5^{\infty} = \frac{1}{50},$$

Thus, if we take $\zeta(3)$ as $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} \approx 1.186$, the error is no more than 1.7%.

5. Conclusion

Thus, we have shown that:

$$f(3) = \int_{0}^{\infty} \frac{x^3}{e^x - 1} dx = \Gamma(4)\zeta(4) = 3!\frac{\pi^4}{90} = \frac{\pi^4}{15}$$

$$f(2) = \int_{0}^{\infty} \frac{x^2}{e^x - 1} dx =$$

$$= \Gamma(3)\zeta(3) = 2.37 \pm 0.04$$

From this, we find the coefficients in the equations we derived earlier, integrating the Planck function.

The average photon concentration is:

$$\langle \epsilon_\gamma \rangle = \left(\frac{\Gamma(4)\zeta(4)}{\Gamma(3)\zeta(3)} \right) kT \approx 2.7kT$$

Stefan-Boltzmann's law:

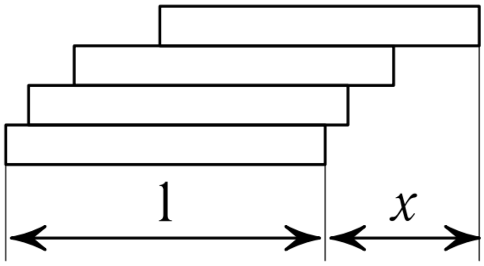
$$F = \left(\pi \frac{2k^4}{c^2 h^3} \Gamma(4)\zeta(4) \right) T^4 =$$

$$= \frac{2k^4 \pi^5}{15 c^2 h^3} T^4 \approx 5.67 \cdot 10^{-8} T^4$$

6. Appendix

The article used the concept of the zeta function, think about what $\zeta(1)$ is. After thinking, try to solve the following two problems:

Problem 1. By stacking several identical plates (e.g., dominoes) on top of each other in the right way, you can form an overhang with a length of x dominoes. What is the maximum possible overhang length x?



Problem 2. You are holding one end of an elastic cord that is 1 km long. From the other end, which is fixed, a beetle is crawling toward you at a speed of 1 cm/s. Each time it crawls 1 cm, you extend the cord by 1 km. The cord is stretched evenly at a constant rate. Will the beetle reach your hand? If so, approximately how long will it take?

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References

[1] Euler "Introductio in analysin infinitorum" 1748
[2] Planck, M. (1901). *On the Law of Distribution of Energy in the Normal Spectrum.* Annalen der Physik.
[3] Born, M., & Wolf, E. (1999). *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light.* Cambridge University Press.
[4] Titchmarsh, E.C. (1986). *The Theory of the Riemann Zeta-Function.* Clarendon Press.
[5] Landau, L. D., Lifshitz, E. M. (1976). *Theoretical Physics. Volume 5. Statistical Physics.* Science.
[6] Kudryavtsev, L. D. (1975). *Course of Mathematical Analysis.* Science.

[7] Gelfand, I. M., Shilov, G. E. (1959). *Generalized functions and actions on them.* Fizmatlit.
[8] Kolmogorov, A. N., Fomin, S. V. (1976). *Elements of the theory of functions and functional analysis.* Science.
[9] Weierstrass, K. (1894). *Zur Theorie der Potenzreihen.*
[10] Apostol, T. M. (1974). *Mathematical Analysis.* Addison-Wesley.